REPORT 977

FREQUENCY RESPONSE OF LINEAR SYSTEMS FROM TRANSIENT DATA

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SUMMARY

Methods are presented that use general correlative time-response input and output data for a linear system to determine the frequency-response function of that system. These methods give an exact description of any linear system for which such transient data are available.

Examples are shown of application of a method to both an underdamped and a critically damped exact second-order system, and to an exact first-order system with and without dead time. Experimental data for a turbine-propeller engine showing the response of engine speed to change in propeller-blade angle are presented and analyzed.

INTRODUCTION

A basic problem confronting the control designer is that of determining the behavior of the controlled system under varying conditions of operation. This question becomes particularly acute when various system parameters, for example, engine shaft torque or turbine-inlet temperature, must be closely controlled in order to prevent system damage or even failure. The problem then becomes one of matching transient behavior of the control to that of the uncontrolled system in order to attain a desired response of the controlled system.

General methods do not exist for the solution of the equations of motion of nonlinear systems. As a result, the analysis of such systems may be impracticably difficult, if, indeed, a solution can be found at all. Because the behavior of many nonlinear systems may be satisfactorily approximated by the assumption of system linearity and because general mathematical methods and techniques for the analysis of linear systems are readily available and (in comparison with present nonlinear methods) relatively simple to apply, the assumption is generally made that the system being studied is linear. The methods of this report are based on such an assumption.

Approaches utilized in the analysis of linear systems (reference 1, pp. 17–18) are: (1) transient analysis, in which the characteristic time response of the system is determined for standard inputs such as step and impulse functions, or (2) frequency analysis, in which system behavior is characterized by the steady-state amplitude and phase relations of the system input and output for sinusoidal inputs of various frequencies.

For inputs such as step or impulse functions, the inherent system characteristics might be obtained by fitting an equation to the output function and from it deriving the differential equation of the system. If the input and output functions were of any general form, fitting differential equations to the data might still be possible, but for systems of inherently high order the accuracy of such a procedure would be low. In general, use of the time-response form of system description involves dealing with convolution integrals (reference 2, p. 54) when the unit so described is to be combined with other units. For complex systems, this descriptive form is not readily adapted to manipulation.

A linear system is known to be characterized by its steady-state response to all frequencies of sinusoidal inputs. The frequency-response form is useful for the description of linear systems because of the ease with which various system characteristics can be manipulated in dealing with combinations of units. For example, the over-all amplitude ratio of output to input for any frequency of input to a system consisting of several units in series is obtained by a simple multiplication of the amplitude ratios for the several units. A possible difficulty in the determination of the steady-state frequency response is the necessity for maintaining a sinusoidal input of constant amplitude and frequency for a length of time sufficient to insure disappearance of transient effects in the output.

The nature of real physical systems may make actual imposition or measurement of step or impulse inputs impossible; in addition, sinusoidal inputs may prove impracticable. The need thus arises for feasible methods capable of handling data of any general form. Such methods that use general correlative time-response data for system input and output to determine the frequency-response characteristics of the system were developed at the NACA Lewis laboratory in 1948–49 and are presented in this report. These methods give an exact description of any linear system for which such data are available.

Three exact methods of obtaining the frequency-response function of a system are shown for system-equilibrium final conditions. A graphical approximation to one of the methods is also given. Modification of one of the exact methods to account for oscillatory final conditions is shown and treatment of dead time is presented. As an illustration of one of the methods, three examples are given, two based on analytically determined system-response curves and one on experimental data obtained at this laboratory for a turbine-propeller engine.

SYMBOLS

The following symbols are used in this report:

A	amplitude of system steady oscillation
a,b	constants
$F(i\omega)$	system frequency-response function
$F(p)$	system transfer function
$f(t)$	general function of time
$G(p)$	system transfer function including dead time
$g(t)$	polynomial approximation to $y(t)$
$h(t)$	normalized Gaussian error distribution or probability pulse
j	order of derivative of $u(t)$
L	Laplace transform
p	complex number
q	degree plus one of $g_k(t)$
R	amplitude ratio of $F(i\omega)$
t	time, seconds
$u(t)$	unit step function
$u'(t)$	unit impulse function
x	general time-dependent input
y	general time-dependent output
β	frequency of system steady oscillation, radians per second
Δt	system dead time, seconds
ζ	damping ratio
θ	phase angle of $F(i\omega)$, radians
σ	standard deviation of Gaussian curve
τ	system time constant, seconds
ϕ	phase angle of system steady oscillation, radians
ω	frequency of system sinusoidal input, radians per second

Subscripts:

0	last term of differential equation
f	final value
k	general term of summation
m	highest order derivative in expression for input
n	highest order derivative in expression for output
r	next to last term of summation

Superscripts:

$','',''',m,n$	first, second, third, mth, and nth time derivatives, respectively

ANALYSIS

The system considered in the following derivation is assumed to be linear. In particular, behavior of the system is assumed representable by a linear differential equation with constant coefficients. The block diagram of a system having an input x and a corresponding output y is shown in figure 1. The symbol $F(p)$ inside the box represents the system operator which, acting on the input x, yields the output y. The linear differential equation relating x and y may be written as

$$a_n\frac{d^ny}{dt^n}+a_{n-1}\frac{d^{n-1}y}{dt^{n-1}}+\cdots+a_0y$$
$$=b_m\frac{d^mx}{dt^m}+b_{m-1}\frac{d^{m-1}x}{dt^{m-1}}+\cdots+b_0x \qquad (1)$$

If the system is initially at rest (or in equilibrium), termwise application to equation (1) of the Laplace transformation (reference 2, pp. 51–56), defined as

$$L[f(t)]=\int_0^\infty f(t)e^{-pt}\,dt \qquad (2)$$

and factoring of the result give

$$(a_np^n+a_{n-1}p^{n-1}+\ldots+a_0)L(y)$$
$$=(b_mp^m+b_{m-1}p^{m-1}+\ldots+b_0)L(x)$$

or

$$L(y)=\frac{b_m\,p^m+b_{m-1}\,p^{m-1}+\ldots+b_0}{a_n\,p^n+a_{n-1}\,p^{n-1}+\ldots+a_0}L(x) \qquad (3)$$

Equation (3) may be formally written as

$$L(y)=F(p)L(x) \qquad (4)$$

where

$$F(p)=\frac{b_m\,p^m+b_{m-1}\,p^{m-1}+\ldots+b_0}{a_n\,p^n+a_{n-1}\,p^{n-1}+\ldots+a_0}$$

The transfer function of the system $F(p)$ is defined as the ratio of the Laplace transform of any normal response of the system to the Laplace transform of the input producing that response. Normal response is the response of the system when initially at rest (reference 2, p. 26).

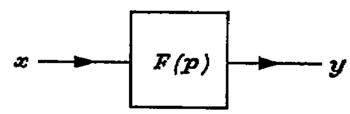


FIGURE 1.—Block diagram of linear system.

From equations (2) and (4), the system transfer function may be written as

$$F(p)=\frac{L(y)}{L(x)}$$

or

$$F(p)=\frac{\int_0^\infty ye^{-pt}\,dt}{\int_0^\infty xe^{-pt}\,dt} \qquad (5)$$

EQUILIBRIUM FINAL CONDITIONS

Derivation of frequency-response function.—The frequency-response function $F(i\omega)$ is formally obtained directly from equation (5) by replacing p by $i\omega$ (reference 1, pp. 96–98). Then, for a sinusoidal input x of frequency ω, the output y ultimately is sinusoidal at the same frequency but with a relative magnitude and phase angle equal to the magnitude and the phase angle of the complex number $F(i\omega)$. The frequency-response function is defined as

$$F(i\omega)=\frac{\int_0^\infty ye^{-i\omega t}\,dt}{\int_0^\infty xe^{-i\omega t}\,dt} \qquad (6)$$

The term $e^{-i\omega t}$ is oscillatory and the integrals of equation (6) may not converge unless $\int_0^\infty |y|\,dt$ and $\int_0^\infty |x|dt$ converge. These conditions unduly restrict the choice of functions that may be used in equation (6) because of the implied requirement that x and y vanish as t increases without limit.

As will be shown, the previous restriction on x and y may be removed by suitable modification of equation (6). A much wider selection of input and output functions is thus permitted.

For a system initially at rest,

$$L[f'(t)]=pL[f(t)] \tag{7}$$

An alternate expression for the transfer function then is

$$F(p)=\frac{L(y')}{L(x')} \tag{8}$$

or, from equations (2) and (8),

$$F(p)=\frac{\int_0^\infty y'e^{-pt}\,dt}{\int_0^\infty x'e^{-pt}\,dt}$$

The frequency-response function then becomes

$$F(i\omega)=\frac{\int_0^\infty y'e^{-i\omega t}\,dt}{\int_0^\infty x'e^{-i\omega t}\,dt} \tag{9}$$

when p is replaced by $i\omega$.

For any input or output function ending in equilibrium, $x'\rightarrow 0$ and $y'\rightarrow 0$, respectively, for $t\rightarrow\infty$. The integrals of equation (9) therefore converge.

From Euler's relation,

$$e^{-i\omega t}=\cos\,\omega t-i\,\sin\,\omega t$$

equation (9) may be written

$$F(i\omega)=\frac{\int_0^\infty y'\,\cos\,\omega t\,dt-i\int_0^\infty y'\,\sin\,\omega t\,dt}{\int_0^\infty x'\,\cos\,\omega t\,dt-i\int_0^\infty x'\,\sin\,\omega t\,dt} \tag{10}$$

This form of the frequency-response function can be used with any data for which the input and the output begin and end in a steady state.

The integrals of equation (10) can be evaluated in various ways. One theoretically exact method, which has been used at the NACA Lewis laboratory, utilizes a rolling-sphere harmonic analyzer. This type of analyzer, the same in operating principle as the device described in reference 3, was designed to obtain the Fourier coefficients from cyclic data, but the manner in which the coefficients are determined results in evaluation of integrals of exactly the form of those in equation (10). The frequency-response function can thus be obtained for any frequency by operating the analyzer over the output curve to evaluate the integrals in the numerator and over the input curve to determine the integrals in the denominator of equation (10) without replotting the data in any other form.

The frequency spectra for nonperiodic phenomena are determined in reference 4 by evaluating Fourier integrals with a rolling-sphere harmonic analyzer in a manner similar to that followed in this report in connection with equation (10). The method of reference 4, however, differs from that of this report in being restricted to functions that begin and end at zero. In addition, the concepts of a system transfer or a system frequency-response function are untreated.

Modifications of frequency-response derivation.—The frequency response may, at times, be desired directly in terms of the input and output functions themselves rather than their time derivatives. For example, if an electronic device were used for analysis, the avoidance of the added complexity and attendant inaccuracy of differentiating circuits might be desirable.

If both x and y come to a steady state by or before some time t_f, equation (9) may be rewritten as

$$F(i\omega)=\frac{\int_0^{t_f} y'e^{-i\omega t}\,dt}{\int_0^{t_f} x'e^{-i\omega t}\,dt} \tag{11}$$

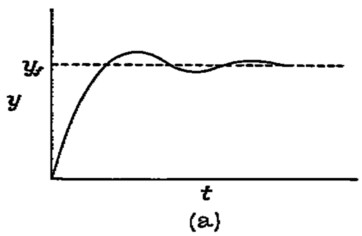
because for $t\geq t_f$ the integrands of equation (11) are zero. Integrating equation (11) by parts gives

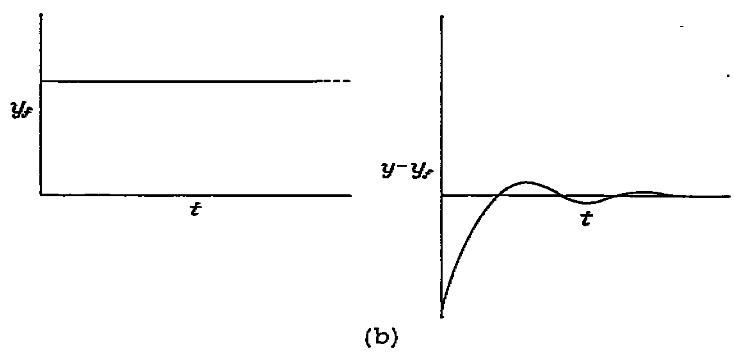
$$F(i\omega)=\frac{i\omega\int_0^{t_f} ye^{-i\omega t}\,dt+ye^{-i\omega t}\Big|_0^{t_f}}{i\omega\int_0^{t_f} xe^{-i\omega t}\,dt+xe^{-i\omega t}\Big|_0^{t_f}}$$

or

$$F(i\omega)=\frac{\left(\int_0^{t_f} y\cos\omega t\,dt-y_f\frac{\sin\omega t_f}{\omega}\right)-i\left(\int_0^{t_f} y\sin\omega t\,dt+y_f\frac{\cos\omega t_f}{\omega}\right)}{\left(\int_0^{t_f} x\cos\omega t\,dt-x_f\frac{\sin\omega t_f}{\omega}\right)-i\left(\int_0^{t_f} x\sin\omega t\,dt+x_f\frac{\cos\omega t_f}{\omega}\right)} \tag{12}$$

An alternate procedure, leading to a somewhat simpler expression, employs the principle of superposition. For example, y may be considered the linear combination of a step function whose ordinate is y_f and a second function





(a) System-output time function.
(b) Components of system-output time function.

FIGURE 2.—Separation of system-output time function into components for equilibrium final conditions.

whose ordinates are those of y displaced downward by the value of y_f. Such a separation of y into its component functions is shown in figure 2. An analytical expression of the process may be derived as follows: The transfer function of the system is

$$F(p)=\frac{\int_0^\infty ye^{-pt}\,dt}{\int_0^\infty xe^{-pt}\,dt} \tag{5}$$

Because of the linearity of the transformation

$$F(p)=\frac{\int_0^\infty (y-y_f)e^{-pt}\,dt+\int_0^\infty y_f e^{-pt}\,dt}{\int_0^\infty (x-x_f)e^{-pt}\,dt+\int_0^\infty x_f e^{-pt}\,dt}$$

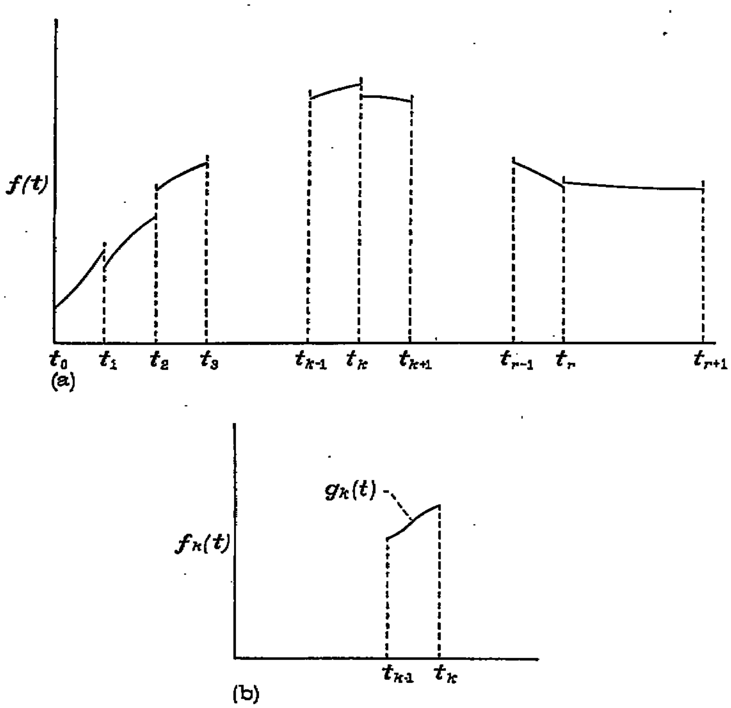
$$=\frac{\int_0^\infty (y-y_f)e^{-pt}\,dt+\frac{y_f}{p}}{\int_0^\infty (x-x_f)e^{-pt}\,dt+\frac{x_f}{p}}$$

Setting $p=i\omega$ then gives

$$F(i\omega)=\frac{\int_0^\infty (y-y_f)\,e^{-i\omega t}\,dt+\frac{y_f}{i\omega}}{\int_0^\infty (x-x_f)\,e^{-i\omega t}\,dt+\frac{x_f}{i\omega}}$$

or

$$F(i\omega)=\frac{\int_0^\infty (y-y_f)\cos\omega t\,dt-i\left[\int_0^\infty (y-y_f)\sin\omega t\,dt+\frac{y_f}{\omega}\right]}{\int_0^\infty (x-x_f)\cos\omega t\,dt-i\left[\int_0^\infty (x-x_f)\sin\omega t\,dt+\frac{x_f}{\omega}\right]} \tag{13}$$



(a) Piecewise continuous time function.
(b) Piecewise approximation to time function.

FIGURE 3.—General approximation to time function.

Equations (12) and (13) define $F(i\omega)$ for all frequencies except $\omega=0$ and $F(0)$ can readily be shown to be y_f/x_f.

Approximation to frequency-response function.—Let $f(t)$ be a piecewise continuous function as shown in figure 3. If within each of the $r+1$ pieces $f(t)$ is approximated by a polynomial of any degree not greater than $q-1$, then, for t_{r+1} at infinity, the Laplace transform of $f(t)$ is given by

$$L[f(t)]=\sum_{k=0}^{r}\sum_{j=1}^{q}\frac{e^{-pt_k}}{p^{q-j+1}}\left[f^{(q-j)}(t_k+0)-f^{(q-j)}(t_k-0)\right] \tag{A10}$$

Equation (A10) is developed in detail in appendix A.

The expression for $L(f)$ in equation (A10) requires only the values of the approximation and its derivatives at $t=+0$ [$f(+0)$, $f'(+0)$, and so forth] together with the jumps $[f^{(q-j)}(t_k+0)-f^{(q-j)}(t_k-0)]$ in f and in the derivatives of the approximating polynomials at any points of discontinuity (t_k) of the functions or their derivatives. The degree of the approximating polynomials may be different in the various pieces but $q-1$ is the highest degree of polynomial in any piece.

As an example of the use of equation (A10), let $f(t)$ be a continuous function (except at $t=0$) approximated, as shown in figure 4, by a series of straight lines. Here the poly-

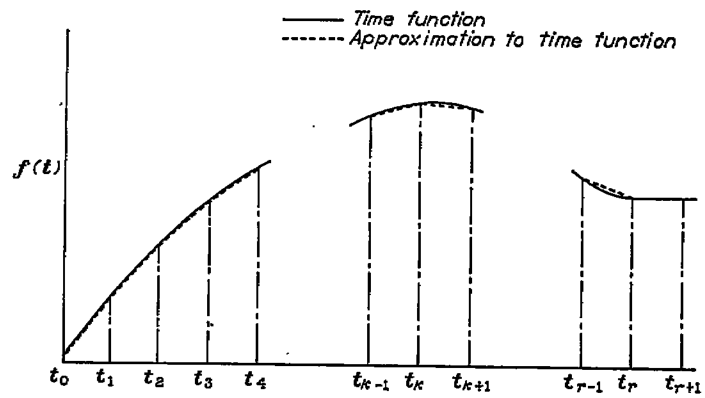


FIGURE 4.—Linear approximation to time function.

nomials are of first degree and $q=2$. Hence, equation (A10) becomes

$$L(f)=\sum_{k=0}^{r}\sum_{j=1}^{2}\frac{e^{-pt_k}}{p^{3-j}}\left[f^{(2-j)}(t_k+0)-f^{(2-j)}(t_k-0)\right] \tag{14}$$

Because f is continuous except at $t=0$

$$f(t_k+0)-f(t_k-0)=0 \qquad \text{for } k>0$$

$$f(t_k+0)-f(t_k-0)=f(+0) \quad \text{for } k=0$$

and

$$L(f)=\frac{1}{p^2}\left\{\sum_{k=1}^{r}e^{-pt_k}[f'(t_k+0)-f'(t_k-0)]+f'(+0)\right\}+\frac{1}{p}f(+0)$$

For evenly spaced t's

$$t_{k+1}-t_k=t_1$$

$$f'(t_k+0)=\frac{f_{k+1}-f_k}{t_1}$$

and for $k>0$

$$f'(t_k-0)=\frac{f_k-f_{k-1}}{t_1}$$

with

$$t_1f'(t_r+0)=f_{r+1}-f_r=0$$

Then

$$L(f)=\frac{1}{p^2}\left[\sum_{k=1}^{r}e^{-pt_k}\left(\frac{f_{k+1}+f_{k-1}-2f_k}{t_1}\right)+\frac{f_1-f(+0)}{t_1}+pf(+0)\right]$$

$$=\frac{1}{p^2t_1}\left\{\sum_{k=1}^{r}e^{-pt_k}(f_{k+1}+f_{k-1}-2f_k)+[f_1-f(+0)]+pt_1f(+0)\right\} \tag{15}$$

In order to get an expression that converges for $p=0$, multiply equation (15) by p and use the relation

$$L(f')=pL(f) \tag{7}$$

Then, on setting $p=i\omega$ and separating real and imaginary parts as in equation (9), equations (16) and (17) are obtained, in which the integrals converge for $\omega=0$.

$$\int_{-0}^{\infty}f'\cos\omega t\,dt=\frac{1}{\omega t_1}\sum_{k=1}^{r}[2f_k-(f_{k-1}+f_{k+1})]\sin\omega kt_1+f(+0) \tag{16}$$

$$\int_{-0}^{\infty}f'\sin\omega t\,dt=\frac{1}{\omega t_1}\left\{[f_1-f(+0)]-\sum_{k=1}^{r}[2f_k-(f_{k-1}+f_{k+1})]\cos\omega kt_1\right\} \tag{17}$$

where

$$kt_1=t_k$$

By applying equations (16) and (17) to the input and output curves of any system and using the results in equation (10), the approximate frequency-response function of that system may be determined.

OSCILLATORY FINAL CONDITIONS

The procedures thus far described apply only when the system comes to rest at some new equilibrium condition. If the output does not reach equilibrium but continues to oscillate about some mean value, a modification in the method is necessary in order to account for the oscillatory component.

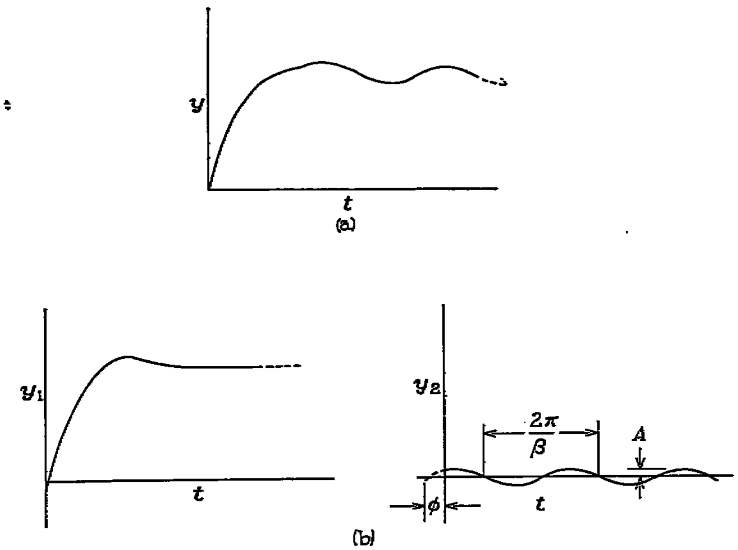
System linearity permits resolution of the output y into two components y_1 and y_2, whose algebraic sum is the original output, as shown in figure 5. The block diagram equivalent to this resolution of y into components is given in figure 6. The dashed outline indicates the over-all transfer function of figure 1. From figure 6, $F(p)$ may be represented as the linear sum of two transfer functions, $F_1(p)$ and $F_2(p)$, operating in parallel. The following equation may then be written:

$$F(p)=F_1(p)+F_2(p)$$

or

$$F(p)=\frac{L(y_1')}{L(x')}+\frac{L(y_2')}{L(x')}$$

$$=\frac{L(y_1')+pL(y_2)}{L(x')} \tag{18}$$



(a) System-output time function.
(b) Components of system-output time function.

FIGURE 5.—Separation of system-output time function into components for oscillatory final conditions.

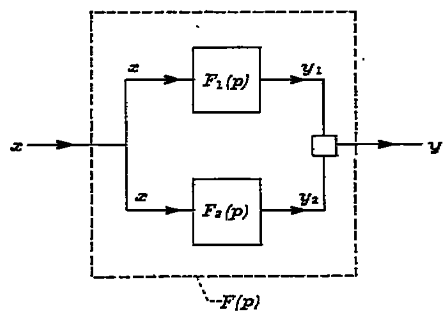


FIGURE 6.—Block-diagram representation of linear system for oscillatory final conditions.

If the steady oscillation is simple harmonic, the oscillatory component of the output may be expressed as

$$y_2=A\sin(\beta t+\phi)$$

for $t>0$. Then

$$pL(y_2)=\frac{Ap}{p^2+\beta^2}(\beta\cos\phi+p\sin\phi)$$

and

$$F(p)=\frac{\int_0^{\infty}y_1'e^{-pt}\,dt+\frac{Ap}{p^2+\beta^2}(\beta\cos\phi+p\sin\phi)}{\int_0^{\infty}x'e^{-pt}\,dt} \tag{19}$$

When p is replaced by $i\omega$ in equation (19), the frequency-response function becomes

$$F(i\omega)=\frac{\left(\int_0^{\infty}y_1'\cos\omega t\,dt-\frac{A\omega^2}{\beta^2-\omega^2}\sin\phi\right)-i\left(\int_0^{\infty}y_1'\sin\omega t\,dt-\frac{A\beta\omega}{\beta^2-\omega^2}\cos\phi\right)}{\int_0^{\infty}x'\cos\omega t\,dt-i\int_0^{\infty}x'\sin\omega t\,dt} \tag{20}$$

TREATMENT OF DEAD TIME

Dead time in a linear system is the time difference between the initiation of a disturbance to the system and the beginning of a response to that disturbance. The mathematical equivalent of dead time then is a translation of the output function along the time axis in a positive direction; that is, if

$$L[y(t)]=F(p)L[x(t)] \tag{4}$$

for a system without dead time, then for the system with dead time

$$L[y(t-\Delta t)]=G(p)L[x(t)] \tag{21}$$

where $G(p)$ is different from $F(p)$. The relation between $G(p)$ and $F(p)$ is readily determined by an expansion in a Taylor's series of $y(t-\Delta t)$ in the neighborhood of t:

$$y(t-\Delta t)=y(t)-\Delta t y'(t)+\frac{\Delta t^2}{2!}y''(t)-\frac{\Delta t^3}{3!}y'''(t)+\ldots$$

Then

$$L[y(t-\Delta t)]=L[y(t)]-\Delta t\,L[y'(t)]+\frac{\Delta t^2}{2!}L[y''(t)]-\frac{\Delta t^3}{3!}L[y'''(t)]+\ldots$$

For initial conditions of equilibrium

$$L[y^n(t)]=p^n L[y(t)]$$

Hence,

$$L[y(t-\Delta t)]=L[y(t)]-\Delta t p L[y(t)]+\frac{\Delta t^2p^2}{2!}L[y(t)]-\frac{\Delta t^3p^3}{3!}L[y(t)]+\ldots$$

$$=\left(1-\Delta tp+\frac{\Delta t^2p^2}{2!}-\frac{\Delta t^3p^3}{3!}+\ldots\right)L[y(t)]$$

$$=e^{-\Delta tp}L[y(t)] \tag{22}$$

From equations (21) and (22)

$$L[y(t)]=e^{\Delta tp}G(p)L[x(t)] \tag{23}$$

Comparison of equations (4) and (23) shows that

$$e^{\Delta tp}G(p)=F(p)$$

or

$$G(p)=e^{-\Delta tp}F(p)$$

from which the corresponding frequency-response functional relation is

$$G(i\omega)=e^{-i\Delta t\omega}F(i\omega) \tag{24}$$

It can easily be shown that the effect of dead time on the frequency-response function is to decrease the phase angle algebraically by the amount $\Delta t\omega$, leaving the amplitude ratio unchanged. In general, $F(i\omega)$ is a complex number and may be expressed as

$$F(i\omega)=Re^{i\theta}$$

From equation (24)

$$G(i\omega)=e^{-i\Delta t\omega}Re^{i\theta}$$

or

$$G(i\omega)=Re^{i(\theta-\Delta t\omega)}$$

EXAMPLES

Examples of the application of the method of this report to determine the frequency-response function of several systems are found in figures 7 to 9. The use of the function thus found has not been covered because the detail with which such use has been treated elsewhere (for example, in references 1 and 2) makes extensive treatment in this report unwarranted.

Examples 1 and 2 illustrate the types of frequency response obtained from several different types of time function. A method of this report was used with the specified time functions to determine the points shown in figures 7 and 8. Frequency-response functions determined directly from the known transfer functions of the systems used were also plotted for comparison. Example 3 presents data for a turbine-propeller engine and shows the type of frequency response obtained, using a method of this report.

EXAMPLE 1

A second-order system having the transfer function

$$F(p)=\frac{1}{p^2+2\zeta p+1} \tag{25}$$

is shown in figure 7 for both the underdamped and critically damped cases. The frequency-response function of the system is obtained directly from equation (25) by replacing p by $i\omega$. For $\zeta=\frac{1}{2}$

$$F(i\omega)=\frac{1}{-\omega^2+i\omega+1}$$

$$=\frac{1-\omega^2}{1-\omega^2+\omega^4}-i\frac{\omega}{1-\omega^2+\omega^4} \tag{26}$$

and for $\zeta=1$

$$F(i\omega)=\frac{1}{-\omega^2+2i\omega+1}$$

$$=\frac{1-\omega^2}{(1+\omega^2)^2}-i\frac{2\omega}{(1+\omega^2)^2} \tag{27}$$

System response to a unit step input is given for $\zeta=\frac{1}{2}$ by

$$y=1-\frac{2\sqrt{3}}{3}e^{-\frac{t}{2}}\sin\left(\frac{\sqrt{3}}{2}t+\frac{\pi}{3}\right) \tag{28}$$

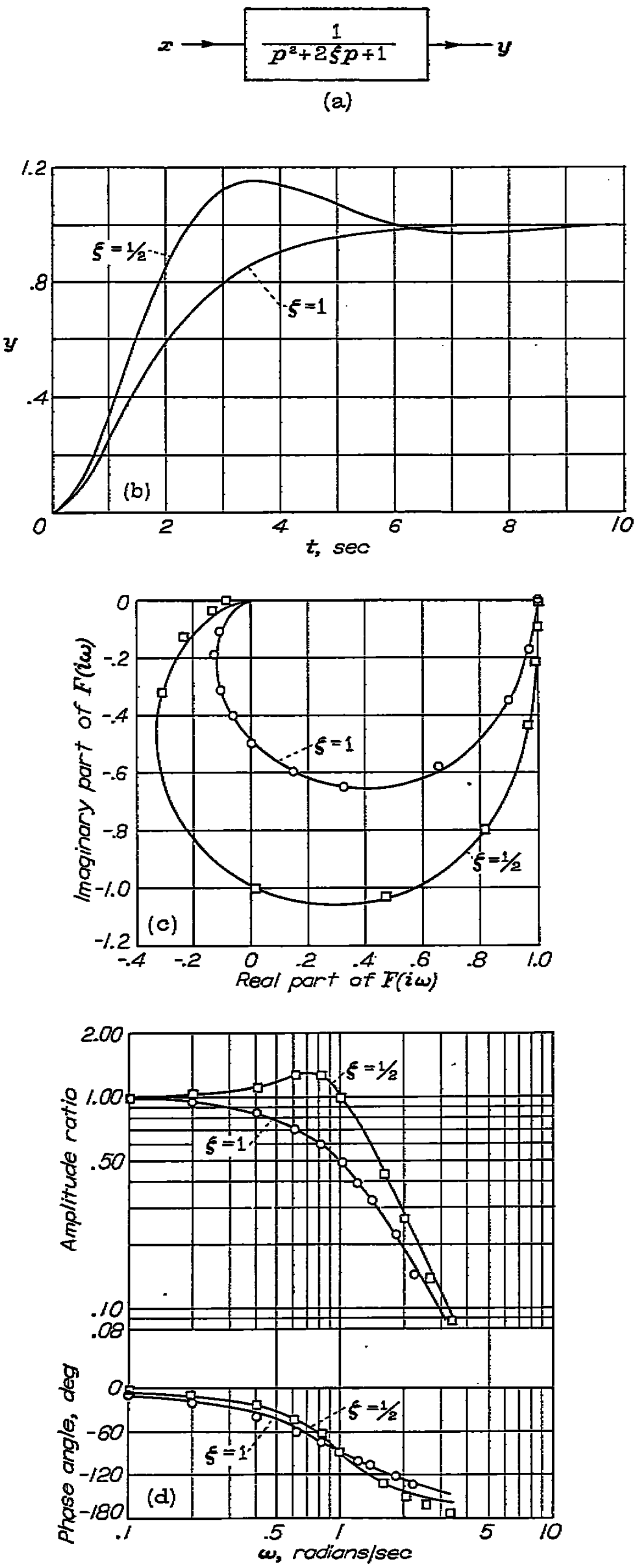
and for $\zeta=1$ by

$$y=1-e^{-t}(1+t) \tag{29}$$

Frequency-response curves calculated from equations (26) and (27) are shown in figures 7 (c) and 7 (d). The rolling-sphere analyzer was used on the transient-response curves determined from equations (28) and (29) to obtain from equation (10) the points shown on the frequency-response curves of figure 7. Because a unit step input was used, the denominator of the right-hand side of equation (10) is equal to 1 for all values of frequency and $F(i\omega)$ can be obtained by operation of the analyzer over the output curves only.

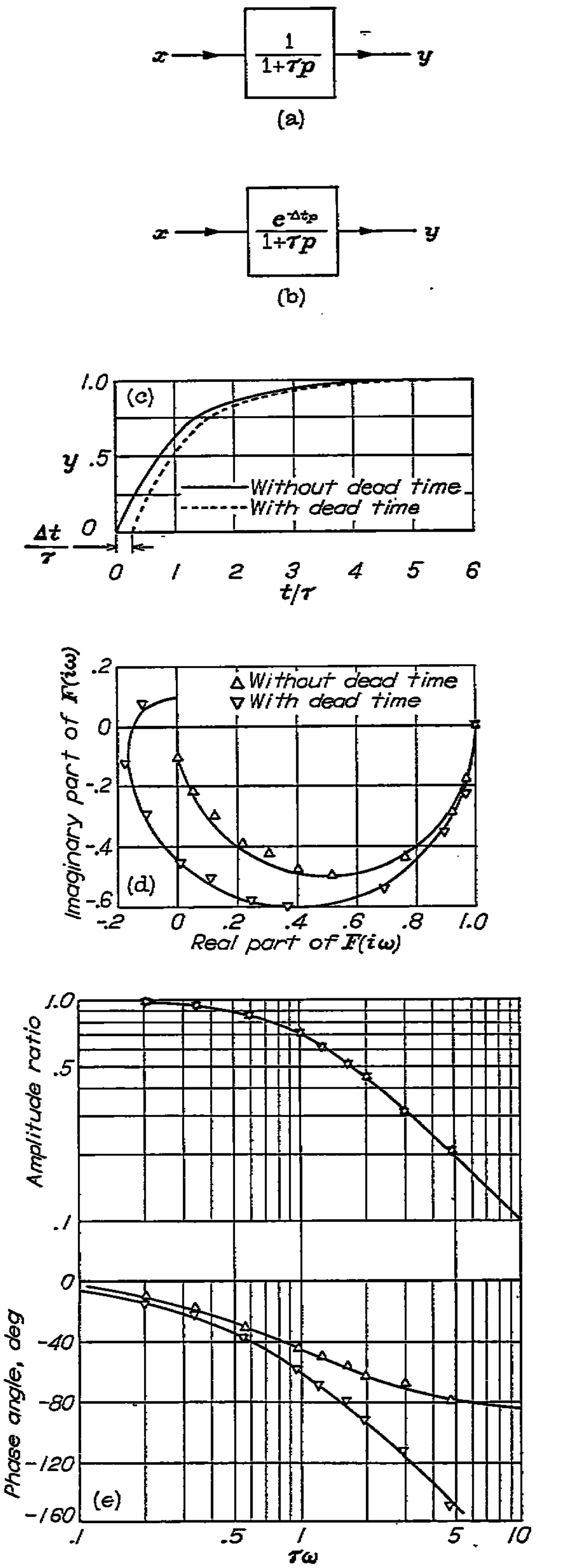
EXAMPLE 2

A first-order system is shown in figure 8. If the system has no dead time, the transfer function is



(a) Second-order system representation.
(b) Response of system to unit step input.
(c) Frequency-response function in complex plane.
(d) Amplitude ratio and phase angle of frequency-response function against ω.

FIGURE 7.—Frequency-response function for second-order system.



(a) Representation of exact first-order system with no dead time.
(b) Representation of exact first-order system with dead time.
(c) System response to unit step input.
(d) Frequency-response function in complex plane.
(e) Amplitude ratio and phase angle of frequency-response function against $\tau\omega$.

FIGURE 8.—Frequency-response function for first-order system.

$$F(p)=\frac{1}{1+\tau p}$$

and the corresponding frequency-response function is

$$F(i\omega)=\frac{1}{1+\tau i\omega}$$

$$=\frac{1}{1+\tau^2\omega^2}-i\,\frac{\tau\omega}{1+\tau^2\omega^2} \tag{31}$$

For a unit step input the output is

$$y=1-e^{-\frac{t}{\tau}} \tag{32}$$

If the system has a dead time Δt, the transfer function becomes

$$G(p)=\frac{e^{-\Delta tp}}{1+\tau p} \tag{33}$$

and

$$G(i\omega)=e^{-i\Delta t\omega}F(i\omega) \tag{34}$$

The response to a unit step input then is

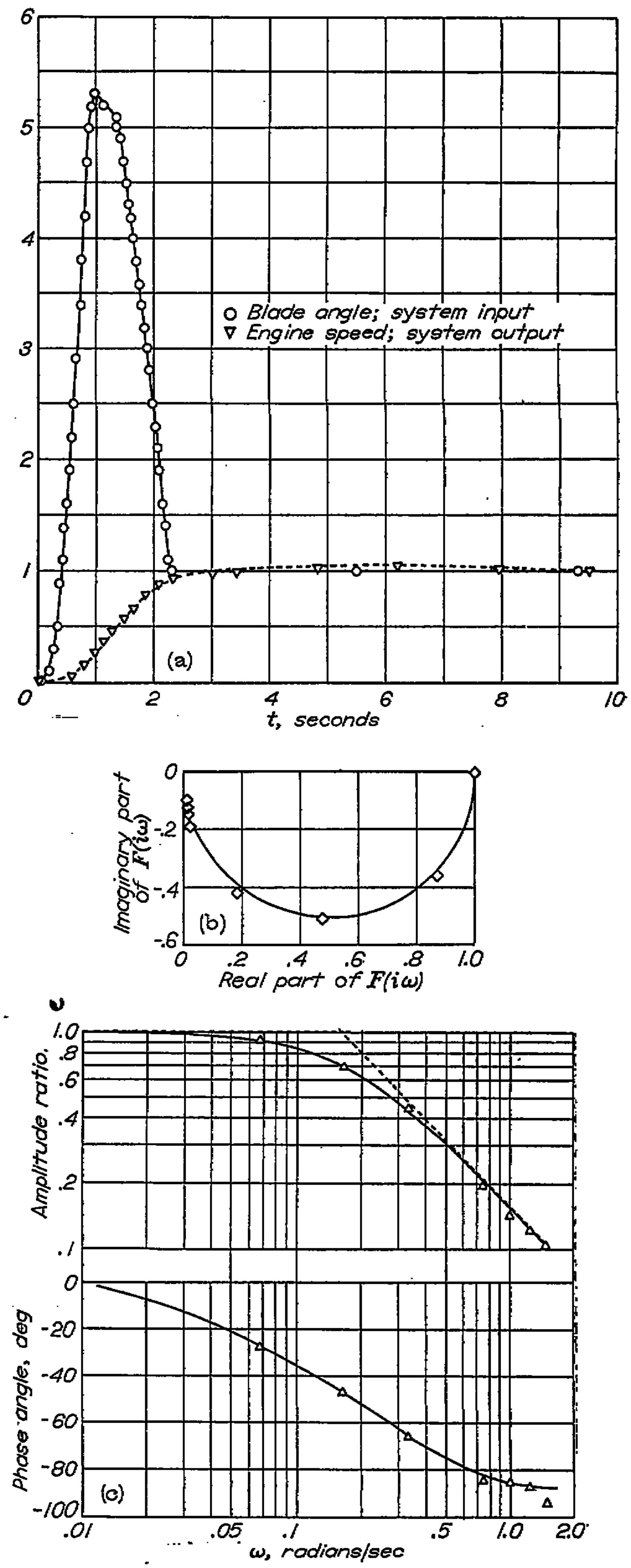
$$y=1-e^{-\frac{t-\Delta t}{\tau}} \tag{35}$$

for $t\geq\Delta t$. The frequency-response curves in figures 8 (d) and 8 (e) were calculated from equations (31) and (34). The function $F(i\omega)$ was found from equation (31) and $G(i\omega)$ was then determined from $F(i\omega)$ by increasing the phase angle of $F(i\omega)$ negatively by the amount $\Delta t\omega$, leaving $|F(i\omega)|$ unchanged. The rolling-sphere analyzer was used on the output curves of equations (32) and (35) to obtain from equation (10) the points shown on the frequency-response curves of figure 8. The use of a unit step input to the system makes the denominator of the right-hand side of equation (10) equal to 1 for all values of frequency. The frequency-response function can therefore be obtained by operation of the analyzer over the output curves only.

EXAMPLE 3

Experimental transient data obtained at this laboratory for the speed response of a turbine-propeller engine to changes in propeller-blade angle are shown in figure 9 (a). Because the frequency-response function is expressed as a ratio, units of the variables involved are unimportant and the ordinate of the data curves has been calibrated in relative values, the difference between initial and final values of the curves being set equal to unity.

The frequency-response function $F(i\omega)$ of this engine for response of engine speed to changes in propeller-blade angle was found from equation (10). The rolling-sphere analyzer was used on the data. In figure 9 (b), $F(i\omega)$ is shown as a frequency-locus plot in the complex plane; figure 9 (c) is a plot of amplitude ratio and phase angle against frequency.



(a) Input and output data for turbine-propeller engine.
(b) Frequency-response function of turbine-propeller engine plotted in complex plane.
(c) Amplitude ratio and phase angle of turbine-propeller frequency-response function against ω.

FIGURE 9.—Frequency-response function of turbine-propeller engine.

The closeness with which the data approach a semicircle in the complex plane indicates that the response of the system is substantially first-order. This inference is further borne out by the amplitude-ratio plot. The asymptotes and the solid curve were calculated for an exact first-order system.

SUMMARY OF RESULTS

Methods are presented for the determination of the frequency-response function of any linear system from general correlative time-response data for input and output of that system. Equations were developed for system equilibrium and oscillatory final conditions and for the effect of system dead time on the frequency-response function.

For illustrative purposes, a method of this report was applied to several linear systems for which the frequency-response functions were known.

Experimental data obtained at this laboratory for the speed response of a turbine-propeller engine to change in propeller-blade angle are presented and analyzed.

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APPENDIX A

DEVELOPMENT OF EXPRESSION FOR GENERAL APPROXIMATION TO FREQUENCY-RESPONSE FUNCTION

Let $f(t)$ be a piecewise continuous time function approximated as shown in figure 3. Thus,

$$f(t)=\sum_{k=1}^{r+1} f_k(t) \tag{A1}$$

where

$$f_k(t)=u(t-t_{k-1})u(t_k-t)g_k(t) \tag{A2}$$

and $u(t-t_{k-1})u(t_k-t)$ is a pulse of unit height and duration t_k-t_{k-1}. Use of the pulse for defining $f_k(t)$ in equation (A2) insures vanishing of the function outside the interval $t_{k-1}<t<t_k$. The polynomial $g_k(t)$ may be of any degree not greater than $q-1$ and may be of different degree in different intervals.

As shown in appendix B, the qth derivative of $f_k(t)$ is

$$f_k^{(q)}(t)=u(t-t_{k-1})u(t_k-t)g_k^{(q)}(t)+\sum_{j=1}^{q} u^{(j)}(t-t_{k-1})g_k^{(q-j)}(t_{k-1})-\sum_{j=1}^{q} u^{(j)}(t-t_k)g_k^{(q-j)}(t_k) \tag{A3}$$

Because $g_k(t)$ is of degree $\leq q-1$,

$$g_k^{(q)}(t)=0$$

Then from

$$f^{(q)}(t)=\sum_{k=1}^{r+1} f_k^{(q)}(t) \tag{A4}$$

and equation (A3)

$$f^{(q)}(t)=\sum_{k=1}^{r+1}\sum_{j=1}^{q} u^{(j)}(t-t_{k-1})g_k^{(q-j)}(t_{k-1})-\sum_{k=1}^{r+1}\sum_{j=1}^{q} u^{(j)}(t-t_k)g_k^{(q-j)}(t_k) \tag{A5}$$

$$f^{(q)}(t)=\sum_{k=1}^{r}\sum_{j=1}^{q} u^{(j)}(t-t_k)[g_{k+1}^{(q-j)}(t_k)-g_k^{(q-j)}(t_k)]+\sum_{j=1}^{q} u^{(j)}(t)g_1^{(q-j)}(0)-\sum_{j=1}^{q} u^{(j)}(t-t_{r+1})g_{r+1}^{(q-j)}(t_{r+1}) \tag{A6}$$

The Laplace transform of f may be defined as

$$L(f)=\int_{-0}^{\infty} fe^{-pt}\,dt \tag{A7}$$

Then, under the assumed zero conditions at $t=-0$,

$$\left.\begin{aligned} L[f^{(q)}(t)]&=p^q L[f(t)]\\ L[u^{(j)}(t-t_k)]&=p^{j-1}e^{-pt_k}\\ \text{and, as } t_{r+1}\rightarrow\infty,&\\ L[u^{(j)}(t-t_{r+1})]&\rightarrow 0\end{aligned}\right\} \tag{A8}$$

Let

$$\left.\begin{aligned} g_{k+1}^{(q-j)}(t_k)&=f^{(q-j)}(t_k+0)\\ g_k^{(q-j)}(t_k)&=f^{(q-j)}(t_k-0)\end{aligned}\right\} \tag{A9}$$

with $f^{(q-j)}(-0)=0$ understood. Then equation (A6), when combined with equations (A8) and (A9), becomes

$$L[f(t)]=\sum_{k=0}^{r}\sum_{j=1}^{q}\frac{e^{-pt_k}}{p^{q-j+1}}[f^{(q-j)}(t_k+0)-f^{(q-j)}(t_k-0)] \tag{A10}$$

which is the equation used in the text. This expression for $L[f(t)]$ includes the values of $f(t)$ and its derivatives at $t=+0$ but not at $t=-0$.

Note that if the Laplace transform of f had been defined as

$$L(f)=\int_{+0}^{\infty} fe^{-pt}\,dt \tag{A11}$$

the same answer would have been obtained for $L(f)$ with $f^{(q-j)}(-0)=0$ understood whether such is the case or not. This result necessarily follows because f is not impulsive at $t=0$. When equation (A11) is used,

$$L[f^{(q)}(t)]=p^q L[f(t)]-\sum_{j=1}^{q} p^{j-1}f^{(q-j)}(+0) \tag{A12}$$

and

$$L[u^{(j)}(t)]=0 \quad \text{for } j\geq 1 \tag{A13}$$

APPENDIX B

DEVELOPMENT OF EXPRESSION FOR $f_k^{(q)}(t)$

In appendix A, $f_k(t)$ is defined by

$$f_k(t)=u(t-t_{k-1})u(t_k-t)g_k(t) \qquad \text{(A2)}$$

Assume, for the moment, that the derivatives of a finitely discontinuous function such as the step and an infinitely discontinuous function such as the impulse have meaning. Then, a formal differentiation of equation (A2) gives

$$f_k'(t)=u(t-t_{k-1})u(t_k-t)g_k'(t)+ u'(t-t_{k-1})u(t_k-t)g_k(t)-u(t-t_{k-1})u'(t_k-t)g_k(t) \qquad \text{(B1)}$$

The impulse u' is different from zero only at its discontinuity. Hence,

$$f_k'(t)=u(t-t_{k-1})u(t_k-t)g_k'(t)+u'(t-t_{k-1})g_k(t_{k-1})-u'(t_k-t)g_k(t_k) \qquad \text{(B2)}$$

because

$$u(t_k-t_{k-1})=1$$

Because u' is an even function

$$u'(t_k-t)=u'(t-t_k)$$

and equation (B2) becomes

$$f_k'(t)=u(t-t_{k-1})u(t_k-t)g_k'(t)+u'(t-t_{k-1})g_k(t_{k-1})-u'(t-t_k)g_k(t_k) \qquad \text{(B3)}$$

Differentiation of equation (B3) gives

$$f_k''(t)=u(t-t_{k-1})u(t_k-t)g_k''(t)+u'(t-t_{k-1})u(t_k-t)g_k'(t)- u(t-t_{k-1})u'(t_k-t)g_k'(t)+u''(t-t_{k-1})g_k(t_{k-1})-u''(t-t_k)g_k(t_k) \qquad \text{(B4)}$$

If the reasoning used on equations (B1) and (B2) is applied to equation (B4),

$$f_k''(t)=u(t-t_{k-1})u(t_k-t)g_k''(t)+[u'(t-t_{k-1})g_k'(t_{k-1})+ u''(t-t_{k-1})g_k(t_{k-1})]-[u'(t-t_k)g_k'(t_k)+u''(t-t_k)g_k(t_k)] \qquad \text{(B5)}$$

Continuation of this process shows finally that, in general,

$$f_k^{(q)}(t)=u(t-t_{k-1})u(t_k-t)g_k^{(q)}(t)+\sum_{j=1}^{q} u^{(j)}(t-t_{k-1})g_k^{(q-j)}(t_{k-1})- \sum_{j=1}^{q} u^{(j)}(t-t_k)g_k^{(q-j)}(t_k) \qquad \text{(A3)}$$

An heuristic justification of the procedure for obtaining equation (A3) will now be given.

A commonly used method of defining the single impulse is to begin with a pulse such as that used in equation (A2) and

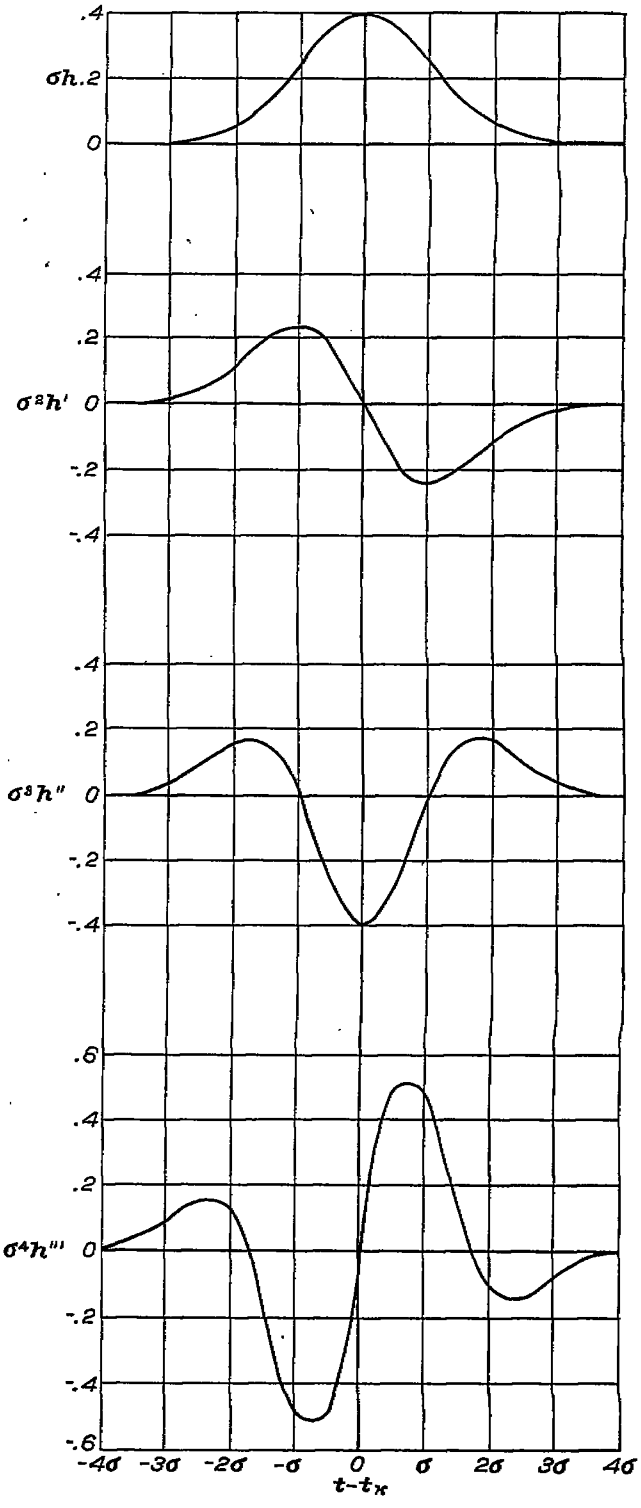

FIGURE 10.—Probability pulse $h(t-t_k)$ and its first three derivatives.

then allow the pulse duration to approach zero while maintaining constant the enclosed area of the pulse (reference 5). Doublet, triplet, and higher-order impulses may be defined similarly but such a means of definition does not lend itself to a consistent interpretation of the meaning of the derivatives of an impulse.

A better means is found in the use of an infinitely differentiable function to define the pulse. Then, the derivatives of the function are the derivatives of the pulse and when, in the limit, the pulse becomes the impulse, the derivatives of the pulse become the derivatives of the impulse. A convenient function is the normalized Gaussian error distribution, defined by

$$h(t-t_k)=\frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{t-t_k}{\sigma}\right)^2} \tag{B6}$$

From

$$\int_{-\infty}^{+\infty} h(t-t_k)\,dt=1 \quad \text{for all } \sigma \tag{B7}$$

and

$$\lim_{\sigma\to 0} h(t-t_k)=0 \qquad (t\neq t_k)$$

$$\lim_{\sigma\to 0} h(t-t_k)=+\infty \qquad (t=t_k)$$

$h(t-t_k)$ is seen to satisfy the requirements for an impulse occurring at $t=t_k$. Equation (B7) holds for $\sigma\to 0$; hence $h(t-t_k)$ satisfies the requirement that the step be its integral as $\sigma\to 0$ or, conversely, that

$$\lim_{\sigma\to 0} h(t-t_k)=u'(t-t_k)$$

By successive differentiations, followed by letting $\sigma\to 0$, it can be seen that

$$u''(t-t_k)=\lim_{\sigma\to 0} h'(t-t_k)$$

$$u'''(t-t_k)=\lim_{\sigma\to 0} h''(t-t_k)$$

and, in general,

$$u^{(j)}(t-t_k)=\lim_{\sigma\to 0} h^{(j-1)}(t-t_k)$$

The function $h(t-t_k)$ and its first three derivatives are shown in figure 10. Note that h and its even derivatives are even functions; the odd derivatives are odd functions.

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